

Elliptic Curves

(PARI-GP version 2.15.3)

An elliptic curve is initially given by 5-tuple $v = [a_1, a_2, a_3, a_4, a_6]$ attached to Weierstrass model or simply $[a_4, a_6]$. It must be converted to an *ell* struct.

Initialize *ell* struct over domain D **E = ellinit**($v, \{D = 1\}$)
over **Q** $D = 1$
over **F_p** $D = p$
over **F_q**, $q = p^f$ $D = \text{ffgen}([p, f])$
over **Q_p**, precision n $D = O(p^n)$
over **C**, current bitprecision $D = 1.0$
over number field K $D = nf$

Points are **[x,y]**, the origin is **[0]**. Struct members accessed as **E.member**:

- All domains: **E.a1,a2,a3,a4,a6, b2,b4,b6,b8, c4,c6, disc, j**
- E defined over **R** or **C**
 - x -coords. of points of order 2 **E.roots**
 - periods / quasi-periods **E.omega, E.eta**
 - volume of complex lattice **E.area**
- E defined over **Q_p**
 - residual characteristic **E.p**
 - If $|j|_p > 1$: Tate's $[u^2, u, q, [a, b], \mathcal{L}]$ **E.tate**
- E defined over **F_q**
 - characteristic **E.p**
 - $\#E(\mathbf{F}_q)/\text{cyclic structure/generators}$ **E.no, E.cyc, E.gen**
- E defined over **Q**
 - generators of $E(\mathbf{Q})$ (require **elldata**) **E.gen**
 - $[a_1, a_2, a_3, a_4, a_6]$ from j -invariant **ellfromj**(j)
 - cubic/quartic/biquadratic to Weierstrass **ellfromeqn**(eq)
 - add points $P + Q$ / $P - Q$ **elladd**(E, P, Q), **ellsub**
 - negate point **ellneg**(E, P)
 - compute $n \cdot P$ **ellmul**(E, P, n)
 - sum of Galois conjugates of P **elltrace**(E, P)
 - check if P is on E **ellisoncurve**(E, P)
 - order of torsion point P **ellorder**(E, P)
 - y -coordinates of point(s) for x **ellordinate**(E, x)
 - $[\wp(z), \wp'(z)] \in E(\mathbf{C})$ attached to $z \in \mathbf{C}$ **ellztopoint**(E, z)
 - $z \in \mathbf{C}$ such that $P = [\wp(z), \wp'(z)]$ **ellpointtoz**(E, P)
 - $z \in \bar{\mathbf{Q}}^*/q^{\mathbf{Z}}$ to $P \in E(\bar{\mathbf{Q}}_p)$ **ellztopoint**(E, z)
 - $P \in E(\bar{\mathbf{Q}}_p)$ to $z \in \bar{\mathbf{Q}}^*/q^{\mathbf{Z}}$ **ellpointtoz**(E, P)
- Change of Weierstrass models, using** $v = [u, r, s, t]$
 - change curve E using v **ellchangecurve**(E, v)
 - change point P using v **ellchangept**(P, v)
 - change point P using inverse of v **ellchangeptinv**(P, v)
- Twists and isogenies**
 - quadratic twist **elltwt**(E, d)
 - n -division polynomial $f_n(x)$ **elldivpol**($E, n, \{x\}$)
 - $[n]P = (\phi_n \psi_n : \omega_n : \psi_n^3)$; return (ϕ_n, ψ_n^2) **ellxn**($E, n, \{x\}$)
 - isogeny from E to E/G **ellisogeny**(E, G)
 - apply isogeny to g (point or isogeny) **ellisogenyapply**(f, g)
 - torsion subgroup with generators **elltors**(E)
- Formal group**
 - formal exponential, n terms **ellformalexp**($E, \{n\}, \{x\}$)
 - formal logarithm, n terms **ellformalog**($E, \{n\}, \{x\}$)
 - $\log_E(-x(P)/y(P)) \in \mathbf{Q}_p$; $P \in E(\mathbf{Q}_p)$ **ellpadiclog**(E, p, n, P)
 - P in the formal group **ellformalpoint**($E, \{n\}, \{x\}$)
 - $[\omega/dt, x\omega/dt]$ **ellformaldifferential**($E, \{n\}, \{x\}$)
 - $w = -1/y$ in parameter $-x/y$ **ellformalw**($E, \{n\}, \{x\}$)

Curves over finite fields, Pairings

random point on E **random**(E)
 $\#E(\mathbf{F}_q)$ **ellcard**(E)
 $\#E(\mathbf{F}_q)$ with almost prime order **ellsea**($E, \{tors\}$)
structure $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$ of $E(\mathbf{F}_q)$ **ellgroup**(E)
is E supersingular? **ellissupersingular**(E)
Weil pairing of m -torsion pts P, Q **ellweilpairing**(E, P, Q, m)
Tate pairing of P, Q ; P m -torsion **elltatepairing**(E, P, Q, m)
Discrete log, find n s.t. $P = [n]Q$ **elllog**($E, P, Q, \{ord\}$)

Curves over Q

Reduction, minimal model

minimal model of E/\mathbf{Q} **ellminimalmodel**($E, \{\&v\}$)
quadratic twist of minimal conductor **ellminimaltwist**(E)
 $[k]P$ with good reduction **ellnonsingularmultiple**(E, P)
 E supersingular at p ? **ellissupersingular**(E, p)
affine points of naïve height $\leq h$ **ellratpoints**(E, h)

Complex heights

canonical height of P **ellheight**(E, P)
canonical bilinear form taken at P, Q **ellheight**(E, P, Q)
height regulator matrix for pts in L **ellheightmatrix**(E, L)

p -adic heights

cyclotomic p -adic height of $P \in E(\mathbf{Q})$ **ellpadicheight**(E, p, n, P)
... bilinear form at $P, Q \in E(\mathbf{Q})$ **ellpadicheight**(E, p, n, P, Q)
... matrix at vector for pts in L **ellpadicheightmatrix**(E, p, n, L)
... regulator for canonical height **ellpadicregulator**(E, p, n, Q)
Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1(E/\mathbf{Q})$ **ellpadicfrobenius**(E, p, n)
slope of unit eigenvector of Frobenius **ellpadics2**(E, p, n)

Isogenous curves

matrix of isogeny degrees for **Q**-isog. curves **ellisomat**(E)
tree of prime degree isogenies **ellisotree**(E)
a modular equation of prime degree N **ellmodulareqn**(N)

L -function

p -th coeff a_p of L -function, p prime **ellap**(E, p)
 k -th coeff a_k of L -function **ellak**(E, k)
 $L(E, s)$ (using less memory than **lfun**) **elllseries**(E, s)
 $L^{(r)}(E, 1)$ (using less memory than **lfun**) **elll1**(E, r)
a Heegner point on E of rank 1 **ellheegner**(E)
order of vanishing at 1 **ellanalyticrank**($E, \{eps\}$)
root number for $L(E, \cdot)$ at p **ellrootno**($E, \{p\}$)
modular parametrization of E **elltaniyama**(E)
degree of modular parametrization **ellmoddegree**(E)
compare with $H^1(X_0(N), \mathbf{Z})$ (for $E' \rightarrow E$) **ellweilcurve**(E)

p -adic L function $L_p^{(r)}(E, d, \chi^s)$ **ellpadicL**($E, p, n, \{s\}, \{r\}, \{d\}$)
BSD conjecture for $L_p^{(r)}(E_D, \chi^0)$ **ellpadicbsd**($E, p, n, \{D = 1\}$)
Iwasawa invariants for $L_p(E_D, \tau^i)$ **ellpadiclamdamu**(E, p, D, i)

Rational points

attempt to compute $E(\mathbf{Q})$ **ellrank**($E, \{\text{effort}\}, \{\text{points}\}$)
initialize for later **ellrank** calls, **ellrankinit**(E)
saturate $\langle P_1, \dots, P_n \rangle$ wrt. primes $\leq B$ **ellsaturation**(E, P, B)
2-covers of the curve E **elll2cover**(E)

Elldata package, Cremona's database:

db code "11a1" \leftrightarrow [*conductor, class, index*] **ellconvertname**(s)
generators of Mordell-Weil group **ellgenerators**(E)
look up E in database **ellidentify**(E)
all curves matching criterion **ellsearch**(N)
loop over curves with cond. from a to b **forell**(E, a, b, seq)

Curves over number field K

coeff a_p of L -function **ellap**(E, p)
Kodaira type of p -fiber of E **elllocalred**(E, p)
integral model of E/K **ellintegralmodel**($E, \{\&v\}$)
minimal model of E/K **ellminimalmodel**($E, \{\&v\}$)
minimal discriminant of E/K **ellminimaldisc**(E)
cond, min mod, Tamagawa num $[N, v, c]$ **ellglobalred**(E)
global Tamagawa number **elltamagawa**(E)
 $P \in E(K)$ n -divisible? $[n]Q = P$ **ellisdivisible**($E, P, n, \{\&Q\}$)
 L -function

A domain $D = [c, w, h]$ in initialization mean we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w, |\Im(s)| < h$; $D = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $D = [1/2, 0, h]$ (critical line up to height h).
vector of first n a_k 's in L -function **ellan**(E, n)
init $L^{(k)}(E, s)$ for $k \leq n$ **L = lfunit**($E, D, \{n = 0\}$)
compute $L(E, s)$ (n -th derivative) **lfun**($L, s, \{n = 0\}$)
 $L(E, 1, r)/(r! \cdot R \cdot \#Sha)$ assuming BSD **ellbsd**(E)

Other curves of small genus

A hyperelliptic curve C is given by a pair $[P, Q]$ ($y^2 + Qy = P$ with $Q^2 + 4P$ squarefree) or a single squarefree polynomial P ($y^2 = P$).
check if $[x, y]$ is on C **hyperellisoncurve**($C, [x, y]$)
discriminant of C **hyperelldisc**(C)
Cremona-Stoll reduction **hyperellred**(C)
apply $m = [e, [a, b; c, d], H]$ to model **hyperellchangecurve**(C, m)
minimal discriminant of integral C **hyperellminimaldisc**(C)
minimal model of integral C **hyperellminimalmodel**(C)
reduction of $y^2 + Qy = P$ (genus 2) **genus2red**($C, \{p\}$)
affine rational points of height $\leq h$ **hyperellratpoints**(C, h)
find a rational point on a conic, ${}^t xGx = 0$ **qfsolve**(G)
 $[H, U]$ such that $H = cU^tGU$ has minimat def **qfminimize**(G)
quadratic Hilbert symbol (at p) **hilbert**($x, y, \{p\}$)
all solutions in \mathbf{Q}^3 of ternary form **qfparam**(G, x)
 $P, Q \in \mathbf{F}_q[X]$; char. poly. of Frobenius **hyperellcharpoly**(Q)
matrix of Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1$ **hyperellpadicfrobenius**

Elliptic & Modular Functions

$w = [\omega_1, \omega_2]$ or *ell* struct (**E.omega**), $\tau = \omega_1/\omega_2$.
arithmetic-geometric mean **agm**(x, y)
elliptic j -function $1/q + 744 + \dots$ **ellj**(x)
Weierstrass $\sigma/\wp/\zeta$ function **ellsigma**(w, z), **ellwp**, **ellzeta**
periods/quasi-periods **ellperiods**($E, \{flag\}$), **ellleta**(w)
 $(2i\pi/\omega_2)^k E_k(\tau)$ **elleisnum**($w, k, \{flag\}$)
modified Dedekind η func. $\prod(1 - q^n)$ **eta**($x, \{flag\}$)
Dedekind sum $s(h, k)$ **sumdedekind**(h, k)
Jacobi sine theta function **theta**(q, z)
 k -th derivative at $z=0$ of $\theta(q, z)$ **thetanullk**(q, k)
Weber's f functions **weber**($x, \{flag\}$)
modular pol. of level N **polmodular**($N, \{inv = j\}$)
Hilbert class polynomial for $\mathbf{Q}(\sqrt{D})$ **polclass**($D, \{inv = j\}$)

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